

The Intersection problem for $2-(v, 5, 1)$ directed block designs

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Abstract

The intersection problem for a pair of $2-(v, 3, 1)$ directed designs and $2-(v, 4, 1)$ directed designs is solved by Fu in 1983 and by Mahmoodian and Soltankhah in 1996, respectively. In this paper we determine the intersection problem for $2-(v, 5, 1)$ directed designs.

KEYWORDS: Directed designs, Intersection of directed designs

1 Introduction

A $t-(v, k, \lambda)$ directed design (or simply a $t-(v, k, \lambda)DD$) is a pair (V, β) , where V is a v -set, and β is a collection of ordered k -tuples of distinct elements of V (blocks), such that each ordered t -tuple of distinct elements of V appears in precisely λ blocks. We say that a t -tuple appears in a k -tuple, if its components appear in that k -tuple as a set, and they appear with the same order. For example, the 5-tuple $(0, 1, 4, 14, 16)$ contains the ordered pairs $(0, 1)$, $(0, 4)$, $(0, 14)$, $(0, 16)$, $(1, 4)$, $(1, 14)$, $(1, 16)$, $(4, 14)$, $(4, 16)$, and $(14, 16)$.

The problem of determining the possible numbers of common blocks between two designs with the same parameters is studied extensively. Kramer and Mesner [14] asked the following: for what values s do there exist two Steiner systems $S(t, k, v)$ intersecting in s blocks? The spectrum of possible intersection sizes for ordinary designs $S(2, 3, v)$ and $S(2, 4, v)$ was settled by Lindner and Rosa [15] and by Colbourn, Hoffman and Lindner [7] respectively. H. L. Fu [9] discussed the intersection numbers of $S(3, 4, v)$ for $v \equiv 4$ or $8 \pmod{12}$, and Hartman and Yehudai [13] completed the determination of the spectrum of possible intersection sizes for Steiner quadruple systems of all admissible orders v except possibly $v = 14, 16$. Lindner and Wallis [16] and independently H. L. Fu [8] settled the spectrum of possible intersection sizes for a pair of $2-(v, 3, 1)DD$ s (transitive triple systems) for all admissible v . The intersection problem for a pair of $2-(v, 4, 1)DD$ s and $3-(v, 4, 1)DD$ s were solved by Mahmoodian and Soltankhah [17, 18]. In this paper, we solve the intersection problem for $2-(v, 5, 1)DD$ s. The existence problem of $2-(v, 5, \lambda)DD$ s has been solved in [23]. The necessary and sufficient condition for the existence of a $2-(v, 5, 1)DD$ is $v \equiv 1$ or $5 \pmod{10}$ with one exception that $2-(15, 5, 1)DD$ does not exist.

The number of blocks in a $2-(v, 5, 1)DD$ is equal to $b_v = \frac{v(v-1)}{10}$. Let $J_D(v) = \{0, 1, \dots, b_v - 2, b_v\}$, and let $I_D(v)$ denote the set of all possible integers m , such that there exist two $2-(v, 5, 1)DD$ s with exactly m common blocks. It is clear that $I_D(v) \subseteq J_D(v)$.

The notation is similar to that used in [19]. Let $K = \{k_1, \dots, k_l\}$ be a set of positive integers. A pairwise balanced design ($PBD(v, K, \lambda)$ or $(K, \lambda)PBD$) of order v with the block sizes from K is a pair (V, β) , where V is a finite set of size v , and β is a family of subsets (blocks) of V , such that (1) if $b \in \beta$ then $|b| \in K$, and (2) every pair of distinct elements of V occurs in exactly λ blocks of β . The notations $PBD(v, K)$ and $K-PBD$ of order v are often used when $\lambda = 1$.

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Let K and G be the sets of positive integers, and let λ be a positive integer. A group divisible design of index λ and order v ($(K, \lambda) - GDD$) is a triple (V, G, β) , where V is a finite set of cardinality v , G is a partition of V into subsets (groups) whose size lie in G , and β is a family of subsets (blocks) of V that satisfy (1) if $b \in \beta$ then $|b| \in K$, and (2) every pair of distinct elements of V occurs in exactly λ blocks or one group, but not both. If $v = a_1g_1 + a_2g_2 + \dots + a_sg_s$ and if there are a_i groups of size g_i , $i = 1, 2, \dots, s$ then the $(K, \lambda) - GDD$ is of the type $g_1^{a_1}g_2^{a_2}\dots g_s^{a_s}$, or is of the type M , where $M = \{g_1, \dots, g_s\}$.

A directed group divisible design ($(K, \lambda) - DGDD$) is a group divisible design GDD in which every block is ordered and each ordered pair formed from distinct elements of different groups occurs in exactly λ blocks.

In this paper, we extensively use the concept of “trade” defined as follows. A $T(t, k, v)$ directed trade of volume s consists of two disjoint collections T_1 and T_2 , each of s blocks, such that each ordered t -tuple occurs in the same number of blocks T_1 as of T_2 . It is usually denoted by $T = (T_1, T_2)$.

Let $D = (V, \beta)$ be a directed design and $T = (T_1, T_2)$ be a $T(v, k, \lambda)$ directed trade of volume s . If $T_1 \subseteq \beta$, we say that D contains the directed trade T , and if we replace T_2 with T_1 , then we obtain a new design $D_1 = (D \setminus T_1) \cup T_2$ which is denoted by $D_1 = D + T$ with the same parameters of D , and $|D_1 \cap D| = b_v - s$. This method of “trade off” are used frequently in this paper.

2 Some small cases

In this section we obtain the intersection size of $2-(v, 5, 1)$ DDs, for $v = 5, 11, 21, 25$ that will be applied in proof of the Theorems A, B and C.

LEMMA 2.1 $I_D(5) = J_D(5)$.

PROOF. Let $D_1 : (0, 1, 2, 3, 4) (4, 3, 2, 1, 0)$ and $D_2 : (1, 0, 2, 3, 4) (4, 3, 2, 0, 1)$ be two $2-(5, 5, 1)$ DDs on the set $\{0, 1, 2, 3, 4\}$. We have $|D_1 \cap D_1| = 2$, $|D_1 \cap D_2| = 0$, so this results in $I_D(5) = J_D(5)$. ■

LEMMA 2.2 $\{0, 1, 2, 3, 11\} \subseteq I_D(11)$.

PROOF. Let D_1 be a $2-(11, 5, 1)$ DD with the base block $(3, 5, 1, 4, 9) \pmod{11}$, and D_2 with the base block $(9, 4, 1, 5, 3) \pmod{11}$ on the set $\{0, 1, \dots, 9, 10\}$, we have $|D_1 \cap D_1| = 11$, $|D_1 \cap D_2| = 0$. Let α denotes a permutation on the same set. For the following permutations on the elements of each block of D_1 and D_2 we have:

$$\begin{aligned} \alpha_1 &= (39)(54) & |D_2 \cap D_1\alpha_1| &= 1; \\ \alpha_2 &= (078) & |D_1 \cap D_1\alpha_2| &= 2; \\ \alpha_3 &= (45) & |D_1 \cap D_1\alpha_3| &= 3. \end{aligned}$$

This results in $\{0, 1, 2, 3, 11\} \subseteq I_D(11)$. ■

LEMMA 2.3 $I_D(21) = J_D(21)$.

PROOF. Let D be a $2-(21, 5, 1)$ DD on the set $\{0, 1, \dots, 20\}$, with the two base blocks $(0, 1, 6, 8, 18)$ and $(1, 0, 16, 14, 4) \pmod{21}$. In design D , there exist 21 disjoint directed trades of volume 2 and at least a directed trade of volume 3:

$$\begin{aligned} T'_i &: (i, 1+i, 6+i, 8+i, 18+i) (1+i, i, 16+i, 14+i, 4+i) \\ T_i & \\ T''_i &: (1+i, i, 6+i, 8+i, 18+i) (i, 1+i, 16+i, 14+i, 4+i) \\ & \quad 0 \leq i \leq 20 \end{aligned}$$

$$\begin{aligned} R' &: (0, 1, 6, 8, 18) (1, 0, 16, 14, 4) (8, 9, 14, 16, 5) \\ R & \\ R'' &: (1, 0, 6, 8, 18) (0, 1, 14, 16, 4) (8, 9, 16, 14, 5) \end{aligned}$$

Let $D_1 = D + R$, so we have:

$$|D \cap D_1| = 39;$$

$$\left| \left(D + \sum_{i=0}^I T_i \right) \cap D \right| = 42 - (2(I+1)) \quad 0 \leq I \leq 20;$$

$$\left| \left(D_1 + \sum_{i=1}^I T_i \right) \cap D \right| = 42 - (2I+3) \quad 1 \leq I \leq 7;$$

$$\left| \left(D_1 + \sum_{i=1}^I T_i \right) \cap D \right| = 42 - (2I+1) \quad 9 \leq I \leq 20.$$

This results in $I_D(21) = J_D(21)$. ■

LEMMA 2.4 $I_D(25) = J_D(25)$.

PROOF. Let D be 2-(25, 5, 1)DD, on the set $V = \mathbb{Z}_5 \times \mathbb{Z}_5$ that is obtained by developing the second coordinate of below base blocks (mod 5) except the first two blocks to which only the first coordinate should be expanded.

$$\begin{array}{ll} ((0, 0), (0, 1), (0, 2), (0, 3), (0, 4)) & ((0, 4), (0, 3), (0, 2), (0, 1), (0, 0)) \\ ((0, 0), (1, 1), (2, 4), (3, 4), (4, 1)) & ((4, 0), (3, 1), (2, 3), (1, 1), (0, 0)) \\ ((0, 3), (1, 2), (2, 3), (3, 1), (4, 1)) & ((4, 0), (3, 3), (2, 2), (1, 2), (0, 3)) \\ ((0, 4), (1, 1), (2, 0), (3, 1), (4, 4)) & ((4, 3), (3, 3), (2, 4), (1, 1), (0, 4)) \\ ((0, 4), (1, 4), (2, 1), (3, 0), (4, 1)) & ((4, 0), (3, 2), (2, 0), (1, 4), (0, 4)) \\ ((0, 3), (1, 1), (2, 1), (3, 3), (4, 2)) & ((4, 1), (3, 0), (2, 0), (1, 1), (0, 3)) \end{array}$$

Now we list some small $T(2, 5, 25)$ directed trades:

$$\begin{array}{l} T_i : ((i, 0), (i, 1), (i, 2), (i, 3), (i, 4))((i, 4), (i, 3), (i, 2), (i, 1), (i, 0)) \\ T_i'' : ((i, 0), (i, 1), (i, 2), (i, 4), (i, 3))((i, 3), (i, 4), (i, 2), (i, 1), (i, 0)) \\ \quad 1 \leq i \leq 5; \\ T_{5+i}' : ((0, i), (1, 1+i), (2, 4+i), (3, 4+i), (4, 1+i))((4, i), (3, 1+i), (2, 3+i), (1, 1+i), (0, i)) \\ T_{5+i}'' : ((1, 1+i), (0, i), (2, 4+i), (3, 4+i), (4, 1+i))((4, i), (3, 1+i), (2, 3+i), (0, i), (1, 1+i)) \\ \quad 1 \leq i \leq 5; \\ T_{10+i}' : ((0, 3+i), (1, 2+i), (2, 3+i), (3, 1+i), (4, 1+i))((4, 0+i), (3, 3+i), (2, 2+i), (1, 2+i), (0, 3+i)) \\ T_{10+i}'' : ((1, 2+i), (0, 3+i), (2, 3+i), (3, 1+i), (4, 1+i))((4, i), (3, 3+i), (2, 2+i), (0, 3+i), (1, 2+i)) \\ \quad 1 \leq i \leq 5; \\ T_{15+i}' : ((0, 4+i), (1, 1+i), (2, i), (3, 1+i), (4, 4+i))((4, 3+i), (3, 3+i), (2, 4+i), (1, 1+i), (0, 4+i)) \\ T_{15+i}'' : ((1, 1+i), (0, 4+i), (2, i), (3, 1+i), (4, 4+i))((4, 3+i), (3, 3+i), (2, 4+i), (0, 4+i), (1, 1+i)) \\ \quad 1 \leq i \leq 5; \\ T_{20+i}' : ((0, 4+i), (1, 4+i), (2, 1+i), (3, i), (4, 1+i))((4, i), (3, 2+i), (2, i), (1, 4+i), (0, 4+i)) \\ T_{20+i}'' : ((1, 4+i), (0, 4+i), (2, 1+i), (3, i), (4, 1+i))((4, i), (3, 2+i), (2, i), (0, 4+i), (1, 4+i)) \\ \quad 1 \leq i \leq 5; \end{array}$$

$$\begin{aligned}
T'_{25+i} &: ((0, 1+i), (1, 4+i), (2, 4+i), (3, 1+i), (4, i))((4, 4+i), (3, 3+i), (2, 3+i), (1, 4+i), (0, 1+i)) \\
T''_{25+i} &: ((1, 4+i), (0, 1+i), (2, 4+i), (3, 1+i), (4, i))((4, 4+i), (3, 3+i), (2, 3+i), (0, 1+i), (1, 4+i)) \\
& \quad 1 \leq i \leq 5.
\end{aligned}$$

$$\begin{aligned}
L' &: ((0, 0), (1, 1), (2, 4), (3, 4), (4, 1))((4, 0), (3, 1), (2, 3), (1, 1), (0, 0)) \\
& \quad ((4, 0), (3, 4), (2, 4), (1, 0), (0, 2)) \\
L &: ((1, 1), (0, 0), (3, 4), (2, 4), (4, 1))((4, 0), (3, 1), (2, 3), (0, 0), (1, 1)) \\
& \quad ((4, 0), (2, 4), (3, 4), (1, 0), (0, 2))
\end{aligned}$$

Let $D_1 = D + L$, so we have:

$$|D \cap D_1| = 57;$$

$$\left| \left(D + \sum_{i=1}^I T_i \right) \cap D \right| = 60 - (2I) \quad 1 \leq I \leq 30;$$

$$\left| \left(D_1 + \sum_{i=2}^I T_i \right) \cap D \right| = 60 - (2(I-1) + 3) \quad 2 \leq I \leq 29.$$

This results in $I_D(25) = J_D(25)$. ■

3 Recursive Construction

In this section we introduce a Construction which will be applied in constructing of directed designs with required intersection sizes.

Construction. If there exists a group divisible design (G, β) of order v with block size 5, 6 and groups each of size congruent to $0, 2 \pmod{5}$, then there exists a $2-(2v+1, 5, 1)$ DD.

PROOF. Let (G, β) be a group divisible design on the element set V . We form a $2-(2v+1, 5, 1)$ DD on the element set $(V \times Z_2) \cup \{\infty\}$ as follows. For each block of size five $b \in \beta$, say $b = (x_1, x_2, x_3, x_4, x_5)$, we form a 5-DGDD of type 2^5 on $b \times Z_2$, such that its groups are $\{x_1\} \times Z_2, \{x_2\} \times Z_2, \{x_3\} \times Z_2, \{x_4\} \times Z_2, \{x_5\} \times Z_2$ and for each block of size six $b \in \beta$, say $b = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, we form a 5-DGDD of type 2^6 on $b \times Z_2$, such that its groups are $\{x_1\} \times Z_2, \{x_2\} \times Z_2, \{x_3\} \times Z_2, \{x_4\} \times Z_2, \{x_5\} \times Z_2, \{x_6\} \times Z_2$. These DGDDs exist, as we will see later on. Finally for each group g of G , we substitute a $2-(2|g|+1, 5, 1)$ DD on $(g \times Z_2) \cup \{\infty\}$. ■

In applying construction and future lemmas we need some DGDDs and GDDs. We may use 5-DGDDs of type 2^5 and 5-DGDDs of type 2^6 . Let $I_G(10)$ be the set of all possible integers m , such that there exist two such 5-DGDDs of type 2^5 with the same groups and exactly m common blocks so let $I_G(12)$ be the set of all possible integers m , such that there exist two such 5-DGDDs of type 2^6 with the same groups and exactly m common blocks.

LEMMA 3.1 $I_G(10) = \{0, 1, \dots, 6, 8\}$.

PROOF. Let G be a 5-DGDD of type 2^5 with the following blocks and groups.
groups:

$$\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{0, 9\}$$

blocks:

$$\begin{array}{cccc}
(7, 9, 6, 4, 2) & (2, 3, 6, 9, 8) & (5, 4, 8, 1, 9) & (2, 4, 7, 0, 5) \\
(6, 0, 7, 3, 1) & (8, 5, 3, 0, 2) & (1, 0, 8, 4, 6) & (9, 1, 3, 5, 7)
\end{array}$$

Now we list some small directed trades:

<u>Directed trades</u>	<u>Blocks removed</u>	<u>Blocks added</u>
T_1	(7, 9, 6, 4, 2) (2, 3, 6, 9, 8)	(7, 6, 9, 4, 2) (2, 3, 9, 6, 8)
T_2	(5, 4, 8, 1, 9) (1, 0, 8, 4, 6)	(5, 8, 4, 1, 9) (1, 0, 4, 8, 6)
T_3	(2, 4, 7, 0, 5) (6, 0, 7, 3, 1)	(2, 4, 0, 7, 5) (6, 7, 0, 3, 1)
T_4	(8, 5, 3, 0, 2) (9, 1, 3, 5, 7)	(8, 3, 5, 0, 2) (9, 1, 5, 3, 7)
T_5	(8, 5, 3, 0, 2) (9, 1, 3, 5, 7) (2, 4, 7, 0, 5)	(8, 3, 0, 5, 2) (9, 1, 5, 3, 7) (2, 4, 7, 5, 0)

Let $G_1 = G + T_5$, we have:

$$|G \cap G_1| = 5;$$

$$\left| G \cap \left(G + \sum_{i=1}^I T_i \right) \right| = 8 - (2i) \quad 1 \leq i \leq 4;$$

$$\left| G \cap \left(G_1 + \sum_{i=1}^I T_i \right) \right| = 8 - (2i + 3) \quad 1 \leq i \leq 2.$$

This results in $I_G(10) = \{0, 1, \dots, 6, 8\}$. ■

LEMMA 3.2 $I_G(12) = \{0, 1, \dots, 10, 12\}$.

PROOF. Let G be a 5-DGDD of type 2^6 with the following blocks and groups.
groups:

$$\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}, \{0, 11\}$$

blocks:

$$\begin{array}{cccc} (1, 3, 0, 7, 6) & (2, 4, 11, 8, 5) & (0, 10, 5, 1, 4) & (6, 8, 4, 0, 9) \\ (4, 2, 10, 6, 7) & (8, 6, 1, 10, 11) & (11, 9, 6, 2, 3) & (9, 11, 7, 4, 1) \\ (3, 1, 9, 5, 8) & (7, 5, 2, 9, 0) & (5, 7, 3, 11, 10) & (10, 0, 8, 3, 2) \end{array}$$

Now we list some small directed trades:

<u>Directed trades</u>	<u>Blocks removed</u>	<u>Blocks added</u>
T_1	(1, 3, 0, 7, 6) (3, 1, 9, 5, 8)	(3, 1, 0, 7, 6) (1, 3, 9, 5, 8)
T_2	(4, 2, 10, 6, 7) (2, 4, 11, 8, 5)	(2, 4, 10, 6, 7) (4, 2, 11, 8, 5)
T_3	(8, 6, 1, 10, 11) (6, 8, 4, 0, 9)	(6, 8, 1, 10, 11) (8, 6, 4, 0, 9)
T_4	(7, 5, 2, 9, 0) (5, 7, 3, 11, 10)	(5, 7, 2, 9, 0) (7, 5, 3, 11, 10)
T_5	(0, 10, 5, 1, 4) (10, 0, 8, 3, 2)	(10, 0, 5, 1, 4) (0, 10, 8, 3, 2)
T_6	(11, 9, 6, 2, 3) (9, 11, 7, 4, 1)	(9, 11, 6, 2, 3) (11, 9, 7, 4, 1)
T_7	(11, 9, 6, 2, 3) (9, 11, 7, 4, 1) (0, 10, 5, 1, 4)	(9, 11, 6, 2, 3) (11, 9, 7, 1, 4) (0, 10, 5, 4, 1)

Let $G_1 = G + T_7$, we have:

$$|G \cap G_1| = 9;$$

$$\left| G \cap \left(G + \sum_{i=1}^I T_i \right) \right| = 12 - (2i) \quad 1 \leq i \leq 6;$$

$$\left| G \cap \left(G_1 + \sum_{i=1}^I T_i \right) \right| = 12 - (2i + 3) \quad 1 \leq i \leq 4.$$

This results in $I_G(12) = \{0, 1, \dots, 10, 12\}$. ■

LEMMA 3.3 Let (G, β) be a group divisible design of order v with r blocks of size 5 and s blocks of size 6, and p groups each of size congruent to $0, 2 \pmod{5}$. For $1 \leq i \leq r$, let $a_i \in I_G(10)$; for $1 \leq i \leq s$, let $c_i \in I_G(12)$; for $1 \leq i \leq p$, let $d_i \in I_D(2|g| + 1)$. Then there exist two $2-(2v + 1, 5, 1)$ DDs intersecting in precisely

$$\sum_{j=1}^r a_j + \sum_{j=1}^s c_j + \sum_{i=1}^p d_i$$

blocks.

PROOF. Using construction, take two copies of the same group divisible design (G, β) and construct on them two $2-(2v+1, 5, 1)$ DDs. Corresponding to each block of size 5, say $b_i = \{x_1, x_2, x_3, x_4, x_5\}$, for $1 \leq i \leq r$ place on $b_i \times Z_2$ in the two systems $5 - DGDs$ of type 2^5 having the same groups, and a_i blocks in common. Corresponding to each block of size 6, say $b_i = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, for $1 \leq i \leq s$ place on $b_i \times Z_2$ in the two systems $5 - DGDs$ of type 2^6 having the same groups, and c_i blocks in common and corresponding to each group g of G place two $2-(2|g|+1, 5, 1)$ DDs with d_i blocks in common. ■

We state following Propositions which provide the bases for main results in the next section.

PROPOSITION 3.1 [3] *Let $v \geq 101$, and $v \equiv 1$ or $5 \pmod{20}$, If $v \neq 141$, then $PBD(v, \{5, 25^*\})$ exists.*

PROPOSITION 3.2 [3] *Let $v \geq 85$, and $v \equiv 1$ or $5 \pmod{20}$, If $v \neq 125$, then $PBD(v, \{5, 21^*\})$ exists.*

PROPOSITION 3.3 [12] *Let $u \equiv 0$ or $2 \pmod{5}$, $u \neq 7$, and $M = \{2, 5, 10, 12, 15, 17, 20, 22, 32, 35, 37, 40, 42, 45, 47, 50, 52, 55, 57, 67, 75, 77, 80, 82, 92, 105, 107, 110, 112, 115, 117, 120, 122, 132, 167\}$, then there exists a $\{5, 6\}$ -GDD of type M and order u .*

PROPOSITION 3.4 [5] *There exists a $\{5, 6\}$ -GDD of type 5^n , for all $n \geq 5$, except possibly when $n \in \{7, 8, 10, 16\}$.*

PROPOSITION 3.5 [21] *There exists a $\{5, 6\}$ -GDD of type $5^{4t+1}u$, for $0 \leq u \leq 5t$, for all $t \geq 1$, $t \notin \{2, 17, 23, 32\}$.*

4 Applying Construction

In this section we prove the following main Theorem.

THEOREM A *For $v \equiv 11 \pmod{20}$, $v \neq 11, 31, 71$, $I_D(v) = J_D(v)$.*

PROOF. According to Proposition 3.4 there exists a $\{5, 6\}$ -GDD of type $5^{\frac{v-1}{10}}$ for each of values $v \equiv 11 \pmod{20}$ and $v \neq 11, 31, 71$. We may apply Construction and Lemma 3.3 and the fact that $I_G(10) = \{0, 1, \dots, 6, 8\}$, $I_G(12) = \{0, 1, \dots, 10, 12\}$ and $\{0, 1, 2, 3, 11\} \subseteq I_D(11)$, we can deduce $I_D(v) = J_D(v)$, for $v \equiv 11 \pmod{20}$, $v \neq 11, 31, 71$. ■

THEOREM B *For $v \equiv 1$ or $5 \pmod{20}$, $v \geq 85$, $I_D(v) = J_D(v)$.*

PROOF. For $v \geq 85$, $v \neq 125$, according to Propositions 3.2 there exists a $PBD(v, \{5, 21^*\})$. If we replace the block of size 21 with a $2-(21, 5, 1)$ DD and put a $2-(5, 5, 1)$ DD on each block of size 5, then for all $v \equiv 1$ or $5 \pmod{20}$, $v \geq 85$, $v \neq 125$, we can obtain a $2-(v, 5, 1)$ DD. So according to Propositions 3.1 there exists a $PBD(125, \{5, 25^*\})$ which if we replace the block of size 25 with a $2-(25, 5, 1)$ DD and put a $2-(5, 5, 1)$ DD on each block of size 5, then we can obtain a $2-(125, 5, 1)$ DD. According to the fact that $I_D(21) = J_D(21)$, $I_D(25) = J_D(25)$ and $I_D(5) = J_D(5)$, therefore we can deduce $I_D(v) = J_D(v)$. ■

THEOREM C *For $v \equiv 15 \pmod{20}$, $v \geq 175$ and $v \neq 215, 235, 335$, $I_D(v) = J_D(v)$.*

PROOF. According to Proposition 3.3, for each of aforesaid values there exists a $\{5, 6\}$ -GDD of order $\frac{v-1}{2}$ with groups that their size belong to the set N which $N \subseteq (M \cap Z_{\frac{v-3}{2}})$. Now according to Construction, Lemma 3.3, Theorems A and B and other results that we obtain in Appendix (the fact that $I_G(10) = \{0, 1, \dots, 6, 8\}$, $I_G(12) = \{0, 1, \dots, 10, 12\}$, $\{0, 1, 2, 3, 11\} \subseteq I_D(11)$, $I_D(2n+1) = J_D(2n+1)$ for $n \in N$), we can deduce $I_D(v) = J_D(v)$. ■

Main Theorem.

For $v \equiv 1, 5 \pmod{10}$, $v \neq 11, 15$, $I_D(v) = J_D(v)$ and $\{0, 1, 2, 3, 11\} \subseteq I_D(11)$.

PROOF. According to Theorems A, B and C for $v \equiv 1, 5 \pmod{10}$, $v \neq \{31, 35, 41, 45, 55, 61, 65, 71, 75, 81, 95, 115, 135, 215, 235, 335\}$, $I_D(v) = J_D(v)$. Theorem for the remaining values of v are proved in the Appendix. ■

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5 Appendix

In this section, we construct $2 - (v, 5, 1)DDs$ for remain values $v = 31, 35, 41, 45, 55, 61, 65, 71, 75, 81, 95, 115, 135, 155, 215, 235, 335$, and by method of trade off we can obtain their intersection spectrum.

- $I_D(31) = J_D(31)$.

Let D be a $2 - (31, 5, 1)DD$ on the set $\{0, 1, \dots, 30\}$, with base blocks $(20, 10, 5, 9, 18)$ $(6, 12, 24, 3, 17)$ $(1, 16, 8, 2, 4) \pmod{31}$. (This directed design is obtained from a $2 - (31, 5, 2)BD$ with a suitable ordering on its base blocks [1]). This $2 - (31, 5, 1)DD$ contains 31 disjoint directed trades of volume 2 and 31 disjoint directed trades of volume 3.

Some $T(2, 5, 31)$ directed trades exist in D :

$$\begin{aligned}
 T'_i &: (1 + i, 16 + i, 8 + i, 2 + i, 4 + i)(2 + i, 8 + i, 20 + i, 30 + i, 13 + i) \\
 T_i &: T'_i : (1 + i, 16 + i, 2 + i, 8 + i, 4 + i)(8 + i, 2 + i, 20 + i, 30 + i, 13 + i) \\
 &\quad i = 1, \dots, 31; \\
 R'_i &: (1 + i, 16 + i, 8 + i, 2 + i, 4 + i)(2 + i, 8 + i, 20 + i, 30 + i, 13 + i) \\
 &\quad (30 + i, 20 + i, 15 + i, 19 + i, 28 + i) \\
 R_i &: R''_i : (1 + i, 16 + i, 2 + i, 8 + i, 4 + i)(8 + i, 2 + i, 30 + i, 20 + i, 13 + i) \\
 &\quad (20 + i, 30 + i, 15 + i, 19 + i, 28 + i) \\
 &\quad i = 1, \dots, 31.
 \end{aligned}$$

Let $D_1 = D + R_1$ so we have: $|D \cap D_1| = 90$.

Odd intersections:

$$\begin{aligned}
 \left| D \cap \left(D + \sum_{i=1}^I T_i \right) \right| &= 93 - 2I & 1 \leq I \leq 31; \\
 \left| D \cap \left(D + \sum_{i=1}^{20} R_i + \sum_{i=21}^I T_i \right) \right| &= 93 - (60 + 2(I - 20)) & 21 \leq I \leq 31; \\
 \left| D \cap \left(D + \sum_{i=1}^{28} R_i + \sum_{i=28}^I T_i \right) \right| &= 93 - (84 + 2(I - 28)) & 28 \leq I \leq 31; \\
 \left| D \cap \left(D + \sum_{i=1}^{30} R_i + T_{31} \right) \right| &= 1.
 \end{aligned}$$

Even intersections:

$$\left| D \cap (D_1 + \sum_{i=2}^I T_i) \right| = 90 - 2(I - 1) \quad 2 \leq I \leq 31;$$

$$\left| D \cap (D + \sum_{i=1}^{21} R_i + \sum_{i=21}^I T_i) \right| = 93 - (63 + 2(I - 20)) \quad 21 \leq I \leq 31;$$

$$\left| D \cap (D + \sum_{i=1}^{27} R_i + \sum_{i=27}^I T_i) \right| = 93 - (81 + 2(I - 26)) \quad 27 \leq I \leq 31;$$

$$\left| D \cap (D + \sum_{i=1}^{29} R_i + \sum_{i=29}^I T_i) \right| = 93 - (87 + 2(I - 28)) \quad 29 \leq I \leq 31;$$

$$\left| D \cap (D + \sum_{i=1}^{31} R_i) \right| = 0.$$

- $I_D(35) = J_D(35)$.

Let D be a 2-(35, 5, 1)DD on the set $\{0, 1, \dots, 34\}$, is obtained by developing the following base blocks under the group generated by $(0)(1, 2, \dots, 17)(18, \dots, 34)$,

$$\begin{array}{lll} (31, 2, 9, 30, 34) & (7, 29, 34, 26, 1) & (18, 25, 1, 31, 5) \\ (4, 6, 1, 9, 2) & (24, 14, 3, 26, 34) & (19, 13, 20, 5, 31) \\ (20, 2, 0, 1, 18). \end{array}$$

(This directed design is obtained from a super simple 2-(35, 5, 2)BD with a suitable ordering on its base blocks [11]). In design D there exist 59 disjoint directed trades of volume 2 and at least a directed trade of volume 3.

Some small $TD(2, 5, 35)$ directed trades exist in D :

$$\begin{array}{ll} T_{1i} : (31 + i, 2 + i, 9 + i, 30 + i, 34 + i)(4 + i, 6 + i, 1 + i, 9 + i, 2 + i) & 1 \leq i \leq 17; \\ T_{1i}'' : (31 + i, 9 + i, 2 + i, 30 + i, 34 + i)(4 + i, 6 + i, 1 + i, 2 + i, 9 + i) & \\ T_{2i} : (7 + i, 29 + i, 34 + i, 26 + i, 1 + i)(24 + i, 14 + i, 3 + i, 26 + i, 34 + i) & 1 \leq i \leq 17; \\ T_{2i}'' : (7 + i, 29 + i, 26 + i, 34 + i, 1 + i)(24 + i, 14 + i, 3 + i, 34 + i, 26 + i) & \\ T_{3i} : (18 + i, 25 + i, 1 + i, 31 + i, 5 + i)(19 + i, 13 + i, 20 + i, 5 + i, 31 + i) & 1 \leq i \leq 17; \\ T_{3i}'' : (18 + i, 25 + i, 1 + i, 5 + i, 31 + i)(19 + i, 13 + i, 20 + i, 31 + i, 5 + i) & \\ T_{4i} : (20 + i, 2 + i, 0, 1 + i, 18 + i)(21 + i, 3 + i, 0, 2 + i, 19 + i) & i = 2I, 1 \leq I \leq 8. \\ T_{4i}'' : (20 + i, 0, 2 + i, 1 + i, 18 + i)(21 + i, 3 + i, 2 + i, 0, 19 + i) & \\ K' : (31, 2, 9, 30, 34)(4, 6, 1, 9, 2)(9, 31, 19, 28, 3) & \\ K & \\ K'' : (9, 31, 2, 30, 34)(4, 6, 1, 2, 9)(31, 9, 19, 28, 3) & \end{array}$$

Let α be the following permutation on the set $\{0, 1, \dots, 34\}$,

$$\alpha = \begin{cases} (m, m + 17) & 1 \leq m \leq 17 \\ (m, m - 17) & 18 \leq m \leq 34 \end{cases}$$

so we have $|D \cap D\alpha| = 0$. Therefore by help of directed trades that there exist in design D we can obtain the set of intersections 2-(35, 5, 1)DD and can deduce $I_D(35) = J_D(35)$.

- $I_D(41) = J_D(41)$.

Let D be a $2 - (41, 5, 1)DD$ on the set $V = \{0, 1, \dots, 40\}$, with the below base blocks [1].

$$\begin{array}{ll} (1, 37, 18, 16, 10) & (2, 33, 36, 32, 20) \\ (13, 18, 37, 39, 4) & (33, 2, 40, 3, 15) \end{array}$$

In design D there exist 82 disjoint directed trades of volume 2 and at least a directed trade K of volume 3.

$$\begin{array}{l} T'_{1i} : (1 + i, 37 + i, 18 + i, 16 + i, 10 + i)(13 + i, 18 + i, 37 + i, 39 + i, 4 + i) \\ T_{1i} \\ T''_{1i} : (1 + i, 18 + i, 37 + i, 16 + i, 10 + i)(13 + i, 37 + i, 18 + i, 39 + i, 4 + i) \\ 0 \leq i \leq 40; \end{array}$$

$$\begin{array}{l} T'_{2i} : (2 + i, 33 + i, 36 + i, 32 + i, 20 + i)(33 + i, 2 + i, 40 + i, 3 + i, 15 + i) \\ T_{2i} \\ T''_{2i} : (33 + i, 2 + i, 36 + i, 32 + i, 20 + i)(2 + i, 33 + i, 40 + i, 3 + i, 15 + i) \\ 0 \leq i \leq 40; \end{array}$$

$$\begin{array}{l} K' : (1, 37, 18, 16, 10)(13, 18, 37, 39, 4)(25, 30, 8, 10, 16) \\ K \\ K'' : (1, 18, 37, 10, 16)(13, 37, 18, 39, 4)(25, 30, 8, 16, 10) \end{array}$$

Therefore by help of the above directed trades, we can obtain the set of intersections $2 - (41, 5, 1)DD$ and can deduce $I_D(41) = J_D(41)$.

- $I_D(45) = J_D(45)$.

Let D be a $2 - (45, 5, 1)DD$ on the set $\{0, 1\} \times Z_{22} \cup \{\infty\}$, is obtained by developing the second coordinate of following 9 base blocks (mod 22). (This directed design is obtained from a super simple $2 - (45, 5, 2)BD$ with a suitable ordering on its base blocks [11])

$$\begin{array}{ll} ((0, 4), (1, 9), (1, 8), (1, 0), (1, 2)) & ((1, 4), (0, 0), (1, 13), (1, 8), (1, 9)) \\ ((1, 21), (0, 18), (0, 21), (0, 6), (0, 20)) & ((0, 0), (1, 15), (0, 18), (1, 21), (0, 8)) \\ ((0, 0), (0, 16), (1, 11), (0, 9), (0, 13)) & ((0, 11), (1, 1), (1, 21), (0, 9), (1, 11)) \\ ((0, 6), (1, 0), \infty, (1, 7), (0, 1)) & ((1, 0), (0, 6), (0, 11), (1, 3), (0, 17)) \\ ((1, 10), (1, 18), (0, 0), (0, 1), (1, 7)) & \end{array}$$

Some small $T(2, 5, 45)$ directed trades exist in D :

$$\begin{array}{l} T'_{1i} : ((1, 4 + i), (0, i), (1, 13 + i), (1, 8 + i), (1, 9 + i)) \\ ((0, 4 + i), (1, 9 + i), (1, 8 + i), (1, i), (1, 2 + i)) \\ T_{1i} \\ T''_{1i} : ((1, 4 + i), (0, i), (1, 13 + i), (1, 9 + i), (1, 8 + i)) \\ ((0, 4 + i), (1, 8 + i), (1, 9 + i), (1, i), (1, 2 + i)) \\ 1 \leq i \leq 22; \\ T'_{2i} : ((0, i), (1, 15 + i), (0, 18 + i), (1, 21 + i), (0, 8 + i)) \\ ((1, 21 + i), (0, 18 + i), (0, 21 + i), (0, 6 + i), (0, 20 + i)) \\ T_{2i} \\ T''_{2i} : ((0, i), (1, 15 + i), (1, 21 + i), (0, 18 + i), (0, 8 + i)) \\ ((0, 18 + i), (1, 21 + i), (0, 21 + i), (0, 6 + i), (0, 20 + i)) \\ 1 \leq i \leq 22; \\ T'_{3i} : ((0, i), (0, 16 + i), (1, 11 + i), (0, 9 + i), (0, 13 + i)) \\ ((0, 11 + i), (1, 1 + i), (1, 21 + i), (0, 9 + i), (1, 11 + i)) \\ T_{3i} \\ T''_{3i} : ((0, i), (0, 16 + i), (0, 9 + i), (1, 11 + i), (0, 13 + i)) \\ ((0, 11 + i), (1, 1 + i), (1, 21 + i), (1, 11 + i), (0, 9 + i)) \\ 1 \leq i \leq 22; \end{array}$$

$$\begin{aligned}
& ((0, 6+i), (1, i), \infty, (1, 7+i), (0, 1+i)) \\
R'_i : & ((1, i), (0, 6+i), (0, 11+i), (1, 3+i), (0, 17+i)) \\
& ((1, 10+i), (1, 18+i), (0, i), (0, 1+i), (1, 7+i)) \\
R_i & \\
& ((1, i), (0, 6+i), \infty, (0, 1+i), (1, 7+i)) \\
R''_i : & ((0, 6+i), (1, i), (0, 11+i), (1, 3+i), (0, 17+i)) \\
& ((1, 10+i), (1, 18+i), (0, i), (1, 7+i), (0, 1+i)) \\
& 1 \leq i \leq 22.
\end{aligned}$$

Therefore by help of the above directed trades that there exist in directed design D , we can obtain the set of intersections design D and we can deduce $I_D(45) = J_D(45)$.

- $I_D(55) = J_D(55)$.

Let D be a 2-(55, 5, 1)DD on the set $V = Z_{54} \cup \{\infty\}$, is obtained by developing the below blocks $+2 \pmod{54}$ (This directed design is obtained from a super simple 2-(55, 5, 2)BD with a suitable ordering on its base blocks [2]).

$$\begin{array}{cccc}
(27, 16, \infty, 38, 25) & (2, 16, 7, 31, 22) & (31, 27, 0, 37, 24) & (3, 49, 11, 23, 25) \\
(0, 39, 25, 38, 19) & (25, 10, 3, 28, 7) & (0, 27, 4, 53, 17) & (0, 23, 11, 30, 40) \\
(1, 36, 38, 39, 34) & (0, 7, 28, 16, 8) & (32, 10, 22, 53, 4) &
\end{array}$$

Some small $T(2, 5, 55)$ directed trades exist in D :

$$\begin{aligned}
R'_{1i} : & (27+2i, 16+2i, \infty, 38+2i, 25+2i)(2i, 39+2i, 25+2i, 38+2i, 19+2i) \\
& (1+2i, 36+2i, 38+2i, 39+2i, 34+2i) \\
R_{1i} & \\
R'_{1i} : & (27+2i, 16+2i, \infty, 25+2i, 38+2i)(2i, 38+2i, 39+2i, 25+2i, 19+2i) \\
& (1+2i, 36+2i, 39+2i, 38+2i, 34+2i) \\
& 0 \leq i \leq 26; \\
R'_{2i} : & (2+2i, 16+2i, 7+2i, 31+2i, 22+2i)(25+2i, 10+2i, 3+2i, 28+2i, 7+2i) \\
& (2i, 7+2i, 28+2i, 16+2i, 8+2i) \\
R_{2i} & \\
R''_{2i} : & (2+2i, 7+2i, 16+2i, 31+2i, 22+2i)(25+2i, 10+2i, 3+2i, 7+2i, 28+2i) \\
& (2i, 28+2i, 7+2i, 16+2i, 8+2i) \\
& 0 \leq i \leq 26; \\
R'_{3i} : & (31+2i, 27+2i, 2i, 37+2i, 24+2i)(2i, 27+2i, 4+2i, 53+2i, 17+2i) \\
& (32+2i, 10+2i, 22+2i, 53+2i, 4+2i) \\
R_{3i} & \\
R''_{3i} : & (31+2i, 2i, 27+2i, 37+2i, 24+2i)(27+2i, 2i, 53+2i, 4+2i, 17+2i) \\
& (32+2i, 10+2i, 22+2i, 4+2i, 53+2i) \\
& 0 \leq i \leq 26; \\
T'_i : & (3+2i, 49+2i, 11+2i, 23+2i, 25+2i)(2i, 23+2i, 11+2i, 30+2i, 40+2i) \\
T_i & \\
T''_i : & (3+2i, 49+2i, 23+2i, 11+2i, 25+2i)(2i, 11+2i, 23+2i, 30+2i, 40+2i) \\
& 0 \leq i \leq 26.
\end{aligned}$$

Therefore by help of directed trades that there exist in directed design D , we can obtain the set of intersections design D and we can deduce $I_D(55) = J_D(55)$.

- $I_D(61) = J_D(61)$.

Let D be a 2-(61, 5, 1)DD on the set $V = \{0, 1, \dots, 60\}$, with the below base blocks [1].

$$\begin{array}{ccc}
(0, 4, 23, 9, 45) & (0, 55, 37, 44, 29) & (0, 60, 48, 58, 27) \\
(4, 0, 42, 56, 20) & (55, 0, 18, 11, 26) & (60, 0, 12, 2, 33)
\end{array}$$

In directed design D there exist 183 disjoint directed trades of volume 2 and least a directed trade K of volume 3.

$$\begin{aligned}
T'_{1i} : & (i, 4+i, 23+i, 9+i, 45+i)(4+i, i, 42+i, 56+i, 20+i) \\
T_{1i} & \\
T''_{1i} : & (4+i, i, 23+i, 9+i, 45+i)(i, 4+i, 42+i, 56+i, 20+i) \\
& 0 \leq i \leq 60;
\end{aligned}$$

$$\begin{aligned}
T_{2i}' &: (i, 55 + i, 37 + i, 44 + i, 29 + i)(55 + i, i, 18 + i, 11 + i, 26 + i) \\
T_{2i}'' &: (55 + i, i, 37 + i, 44 + i, 29 + i)(i, 55 + i, 18 + i, 11 + i, 26 + i) \\
&\quad 0 \leq i \leq 60; \\
T_{3i}' &: (i, 60 + i, 48 + i, 58 + i, 27 + i)(60 + i, i, 12 + i, 2 + i, 33 + i) \\
T_{3i}'' &: (60 + i, i, 48 + i, 58 + i, 27 + i)(i, 60 + i, 12 + i, 2 + i, 33 + i) \\
&\quad 0 \leq i \leq 60; \\
K' &: (0, 60, 48, 58, 27)(60, 0, 12, 2, 33)(24, 25, 37, 27, 58) \\
K & \\
K'' &: (60, 0, 48, 27, 58)(0, 60, 12, 2, 33)(24, 25, 37, 58, 27)
\end{aligned}$$

Therefore by help of the above three groups of directed trades, we can obtain the intersections set 2-(61, 5, 1)DD and can deduce $I_D(61) = J_D(61)$.

- $I_D(65) = J_D(65)$.

Let D be a 2-(65, 5, 1)DD on the set $V = \mathbb{Z}_{54} \cup \{\infty_0, \dots, \infty_{10}\}$, that is obtained by developing the below base blocks $+2 \pmod{54}$, that for $z = 0, 3$ replace ∞_z by ∞_{z+x} ($x = 1, 2$) when adding any value $\equiv 2x \pmod{6}$ to a base block. Finally form a 2-(11, 5, 1)DD on the set $\{\infty_0, \dots, \infty_{10}\}$.

$$\begin{array}{ccccc}
(31, 0, 35, 41, 20) & (3, 1, 41, 15, 35) & (4, 9, 2, \infty_3, 27) & (\infty_0, 29, 1, 28, 30) & (41, 8, \infty_6, 29, 0) \\
(0, 31, \infty_3, 8, 12) & (1, 3, 52, 47, \infty_0) & (11, \infty_3, 19, 28, 35) & (12, 25, \infty_7, 28, 1) & (47, 12, \infty_9, 0, 29) \\
(0, 14, \infty_0, 32, 15) & (20, 14, 44, 0, 53) & (0, 44, 28, 50, 47) & (3, 8, \infty_{10}, 53, 44) & (31, 42, \infty_8, 53, 14)
\end{array}$$

(This directed design is obtained from a super simple 2-(65, 5, 2)BD with a suitable ordering on its base blocks [2]). In design D there exist 135 disjoint directed trades of volume 2 and 27 disjoint directed trades of volume 5 and least a directed trade of volume 3.

$$\begin{aligned}
T_{1i}' &: (31 + 2i, 2i, 35 + 2i, 41 + 2i, 20 + 2i)(2i, 31 + 2i, \infty_3, 8 + 2i, 12 + 2i) \\
T_{1i}'' &: (2i, 31 + 2i, 35 + 2i, 41 + 2i, 20 + 2i)(31 + 2i, 2i, \infty_3, 8 + 2i, 12 + 2i) \\
&\quad 1 \leq i \leq 27; \\
T_{2i}' &: (3 + 2i, 1 + 2i, 41 + 2i, 15 + 2i, 35 + 2i)(1 + 2i, 3 + 2i, 52 + 2i, 47 + 2i, \infty_{0+x}) \\
T_{2i}'' &: (1 + 2i, 3 + 2i, 41 + 2i, 15 + 2i, 35 + 2i)(3 + 2i, 1 + 2i, 52 + 2i, 47 + 2i, \infty_{0+x}) \\
&\quad 1 \leq i \leq 27; \\
T_{3i}' &: (30 + 2i, 35 + 2i, 28 + 2i, \infty_{3+x}, 53 + 2i)(11 + 2i, \infty_{3+x}, 19 + 2i, 28 + 2i, 35 + 2i) \\
T_{3i}'' &: (30 + 2i, 28 + 2i, 35 + 2i, \infty_{3+x}, 53 + 2i)(11 + 2i, \infty_{3+x}, 19 + 2i, 35 + 2i, 28 + 2i) \\
&\quad 1 \leq i \leq 27; \\
T_{4i}' &: (\infty_{0+x}, 29 + 2i, 1 + 2i, 28 + 2i, 30)(12 + 2i, 25 + 2i, \infty_7, 28 + 2i, 1 + 2i) \\
T_{4i}'' &: (\infty_{0+x}, 29 + 2i, 28 + 2i, 1 + 2i, 30)(12 + 2i, 25 + 2i, \infty_7, 1 + 2i, 28 + 2i) \\
&\quad 1 \leq i \leq 27; \\
T_{5i}' &: (41 + 2i, 8 + 2i, \infty_6, 29 + 2i, 2i)(47 + 2i, 12 + 2i, \infty_9, 2i, 29 + 2i) \\
T_{5i}'' &: (41 + 2i, 8 + 2i, \infty_6, 2i, 29 + 2i)(47 + 2i, 12 + 2i, \infty_9, 29 + 2i, 2i) \\
&\quad 1 \leq i \leq 27; \\
R_i' &: (2i, 14 + 2i, \infty_{0+x}, 32 + 2i, 15 + 2i)(20 + 2i, 14 + 2i, 44 + 2i, 2i, 53 + 2i) \\
R_i'' &: (2i, 44 + 2i, 28 + 2i, 50 + 2i, 47 + 2i)(3 + 2i, 8 + 2i, \infty_{10}, 53 + 2i, 44 + 2i) \\
&\quad (31 + 2i, 42 + 2i, \infty_8, 53 + 2i, 14 + 2i) \\
R_i' &: (14 + 2i, 2i, \infty_{0+x}, 32 + 2i, 15 + 2i)(20 + 2i, 2i, 53 + 2i, 14 + 2i, 44 + 2i) \\
R_i'' &: (44 + 2i, 2i, 28 + 2i, 50 + 2i, 47 + 2i)(3 + 2i, 8 + 2i, \infty_{10}, 44 + 2i, 53 + 2i) \\
&\quad (31 + 2i, 42 + 2i, \infty_8, 14 + 2i, 53 + 2i) \\
&\quad 1 \leq i \leq 27.
\end{aligned}$$

$$\begin{aligned}
K' &: (0, 14, \infty_0, 32, 15)(20, 14, 44, 0, 53)(0, 44, 28, 50, 47) \\
K & \\
K'' &: (14, 0, \infty_0, 32, 15)(20, 0, 14, 44, 53)(44, 0, 28, 50, 47)
\end{aligned}$$

Therefore according to the six groups of disjoint directed trades that there exist in design D , we can obtain the intersection set of design 2-(65, 5, 1)DD and can deduce $I_D(65) = J_D(65)$.

- $I_D(71) = J_D(71)$.

Let D be a 2-(71, 5, 1)DD on the set $\{0, 1, \dots, 70\}$, with the following base blocks (mod 71).

$$\begin{aligned}
(28, 45, 14, 15, 20) & \quad (20, 4, 15, 29, 3) & (65, 63, 4, 20, 54) \\
(58, 40, 6, 8, 30) & \quad (50, 30, 8, 11, 26) \\
(43, 37, 2, 50, 10) & \quad (50, 2, 35, 6, 70)
\end{aligned}$$

(This directed design is obtained from a 2-(71, 5, 2)BD with a suitable ordering on its base blocks [1]). In design D , there exist 71 disjoint directed trades of volume 3 and 142 disjoint directed trades of volume 2:

$$\begin{aligned}
R_i' &: (28 + i, 45 + i, 14 + i, 15 + i, 20 + i)(20 + i, 4 + i, 15 + i, 29 + i, 3 + i) \\
& \quad (65 + i, 63 + i, 4 + i, 20 + i, 54 + i) \\
R_i & \\
R_i'' &: (28 + i, 45 + i, 14 + i, 20 + i, 15 + i)(4 + i, 15 + i, 20 + i, 29 + i, 3 + i) \\
& \quad (65 + i, 63 + i, 20 + i, 4 + i, 54 + i) \\
& \quad 0 \leq i \leq 70; \\
T_{1i}' &: (58 + i, 40 + i, 6 + i, 8 + i, 30 + i)(50 + i, 30 + i, 8 + i, 11 + i, 26 + i) \\
T_{1i} & \\
T_{1i}'' &: (58 + i, 40 + i, 6 + i, 30 + i, 8 + i)(50 + i, 8 + i, 30 + i, 11 + i, 26 + i) \\
& \quad 0 \leq i \leq 70; \\
T_{2i}' &: (43 + i, 37 + i, 2 + i, 50 + i, 10 + i)(50 + i, 2 + i, 35 + i, 6 + i, 70 + i) \\
T_{2i} & \\
T_{2i}'' &: (43 + i, 37 + i, 50 + i, 2 + i, 10 + i)(2 + i, 50 + i, 35 + i, 6 + i, 70 + i) \\
& \quad 0 \leq i \leq 70.
\end{aligned}$$

Therefore by help of the above three groups of directed trades, we can obtain the intersection set of design D and deduce $I_D(71) = J_D(71)$.

- $I_D(75) = J_D(75)$.

Let D be a 2-(75, 5, 1)DD on the set $V = Z_{74} \cup \{\infty\}$, is obtained by developing the below base blocks +2 (mod 74). (This directed design is obtained from a super simple 2-(75, 5, 2)BD with a suitable ordering on its base blocks [6]).

$$\begin{aligned}
(6, 0, 47, 44, 3) & \quad (40, 65, 26, 3, 44) & (64, 29, 52, 63, 0) & (2, 15, \infty, 59, 0) & (22, 15, 2, 42, 34) \\
(49, 32, 0, 63, 6) & \quad (48, 27, 23, 17, 65) & (17, 23, 5, 60, 56) & (25, 3, 27, 43, 1) & (0, 59, 50, 69, 58) \\
(27, 48, 0, 30, 2) & \quad (46, 1, 6, 61, 70) & (60, 7, 61, 0, 65) & (22, 58, 45, 67, 0) & (1, 2, 37, 9, 29)
\end{aligned}$$

This 2-(75, 5, 1)DD contain 111 disjoint directed trades of volume 2 and 111 disjoint directed trades of volume 3. Now list these $T(2, 5, 75)$ directed trades:

$$\begin{aligned}
T_{1i}' &: (6 + 2i, 2i, 47 + 2i, 44 + 2i, 3 + 2i)(40 + 2i, 65 + 2i, 26 + 2i, 3 + 2i, 44 + 2i) \\
T_{1i} & \\
T_{1i}'' &: (6 + i2, 2i, 47 + 2i, 3 + 2i, 44 + 2i)(40 + 2i, 65 + 2i, 26 + 2i, 44 + 2i, 3 + 2i) \\
& \quad 0 \leq i \leq 36; \\
T_{2i}' &: (64 + 2i, 29 + 2i, 52 + 2i, 63 + 2i, 0)(49 + 2i, 32 + 2i, 2i, 63 + 2i, 6 + 2i) \\
T_{2i} & \\
T_{2i}'' &: (64 + 2i, 29 + 2i, 52 + 2i, 2i, 63 + 2i)(49 + 2i, 32 + 2i, 63 + 2i, 2i, 6 + 2i) \\
& \quad 0 \leq i \leq 36;
\end{aligned}$$

$$\begin{aligned}
T'_{3i} : & (2 + 2i, 15 + 2i, \infty, 59 + 2i, 2i)(22 + 2i, 15 + 2i, 2 + 2i, 42 + 2i, 34 + 2i) \\
T''_{3i} : & (15 + 2i, 2 + 2i, \infty, 59 + 2i, 2i)(22 + 2i, 2 + 2i, 15 + 2i, 42 + 2i, 34 + 2i) \\
& 0 \leq i \leq 36; \\
R'_{1i} : & (27 + 2i, 48 + 2i, 2i, 30 + 2i, 2 + 2i)(48 + 2i, 27 + 2i, 23 + 2i, 17 + 2i, 65 + 2i) \\
& (17 + 2i, 23 + 2i, 5 + 2i, 60 + 2i, 56 + 2i) \\
R''_{1i} : & (48 + 2i, 27 + 2i, 2i, 30 + 2i, 2 + 2i)(27 + 2i, 48 + 2i, 17 + 2i, 23 + 2i, 65 + 2i) \\
& (23 + 2i, 17 + 2i, 5 + 2i, 60 + 2i, 56 + 2i) \\
& 0 \leq i \leq 36; \\
R'_{2i} : & (22 + 2i, 58 + 2i, 45 + 2i, 67 + 2i, 2i)(67 + 2i, 45 + 2i, 69 + 2i, 11 + 2i, 43 + 2i) \\
& (8 + 2i, 67 + 2i, 58 + 2i, 3 + 2i, 66 + 2i) \\
R''_{2i} : & (22 + 2i, 67 + 2i, 58 + 2i, 45 + 2i, 2i)(45 + 2i, 67 + 2i, 69 + 2i, 11 + 2i, 43 + 2i) \\
& (8 + 2i, 58 + 2i, 67 + 2i, 66 + 2i, 3 + 2i) \\
& 0 \leq i \leq 36; \\
R'_{3i} : & (1 + 2i, 2 + 2i, 37 + 2i, 9 + 2i, 29 + 2i)(72 + 2i, 27 + 2i, 32 + 2i, 13 + 2i, 22 + 2i) \\
& (8 + 2i, 29 + 2i, 9 + 2i, 22 + 2i, 13 + 2i) \\
R''_{3i} : & (1 + 2i, 2 + 2i, 37 + 2i, 29 + 2i, 9 + 2i)(72 + 2i, 27 + 2i, 32 + 2i, 22 + 2i, 13 + 2i) \\
& (8 + 2i, 9 + 2i, 29 + 2i, 13 + 2i, 22 + 2i) \\
& 0 \leq i \leq 36.
\end{aligned}$$

Therefore by help of the above six groups of directed trades we can obtain the set of intersections of design D and we can deduce $I_D(75) = J_D(75)$.

- $I_D(81) = J_D(81)$.

Let D be a $2-(81, 5, 1)$ DD on the set $V = \{0, 1, \dots, 80\}$, with the below base blocks [1].

$$\begin{array}{cccc}
(0, 1, 12, 5, 26) & (0, 2, 40, 10, 64) & (0, 3, 47, 18, 53) & (0, 9, 32, 48, 68) \\
(1, 0, 70, 77, 56) & (2, 0, 43, 73, 19) & (3, 0, 37, 66, 31) & (9, 0, 58, 42, 22)
\end{array}$$

In directed design D there exist 324 disjoint directed trades of volume 2 and least a directed trade K of volume 3.

$$\begin{aligned}
T'_{1i} : & (i, 1 + i, 12 + i, 5 + i, 26 + i)(1 + i, i, 70 + i, 77 + i, 56 + i) \\
T''_{1i} : & (1 + i, i, 12 + i, 5 + i, 26 + i)(i, 1 + i, 70 + i, 77 + i, 56 + i) \\
& 0 \leq i \leq 80; \\
T'_{2i} : & (i, 2 + i, 40 + i, 10 + i, 64 + i)(2 + i, i, 43 + i, 73 + i, 19 + i) \\
T''_{2i} : & (2 + i, i, 40 + i, 10 + i, 64 + i)(i, 2 + i, 43 + i, 73 + i, 19 + i) \\
& 0 \leq i \leq 80; \\
T'_{3i} : & (i, 3 + i, 47 + i, 18 + i, 53 + i)(3 + i, i, 37 + i, 66 + i, 31 + i) \\
T''_{3i} : & (3 + i, i, 47 + i, 18 + i, 53 + i)(i, 3 + i, 37 + i, 66 + i, 31 + i) \\
& 0 \leq i \leq 80; \\
T'_{4i} : & (i, 9 + i, 32 + i, 48 + i, 68 + i)(9 + i, i, 58 + i, 42 + i, 22 + i) \\
T''_{4i} : & (9 + i, i, 32 + i, 48 + i, 68 + i)(i, 9 + i, 58 + i, 42 + i, 22 + i) \\
& 0 \leq i \leq 80;
\end{aligned}$$

$$\begin{array}{l}
K' : (0, 1, 12, 5, 26)(1, 0, 70, 77, 56)(65, 66, 77, 70, 10) \\
K \\
K'' : (1, 0, 12, 5, 26)(0, 1, 77, 70, 56)(65, 66, 70, 77, 10) \\
0 \leq i \leq 80.
\end{array}$$

Therefore by help of the above four groups of directed trades, we can obtain the set of intersections of design D and can deduce $I_D(81) = J_D(81)$.

- $I_D(95) = J_D(95)$.

Let D be a 2-(95, 5, 1)DD on the set $V = Z_{94} \cup \{\infty\}$, that is obtained by developing the below base blocks +1 (mod 94). (This directed design is obtained from a super simple 2-(95, 5, 2)BD with a suitable ordering on its base blocks [6]).

$$\begin{array}{cccc}
(4, 79, 61, 1, 74) & (42, 47, 44, 52, 37) & (41, 66, 45, 32, 91) & (3, 72, 40, 52, 9) \\
(85, 35, 74, 1, 58) & (66, 1, 52, 44, 88) & (2, 0, 66, 41, 48) & (19, 52, 40, 0, 90) \\
(20, 37, 48, 54, 21) & (61, 81, 55, 19, 83) & (48, 57, 55, 81, 1) & (81, 57, 5, 84, 17) \\
(1, 83, 48, 37, 66) & (9, 1, 10, 62, 81) & (18, 58, 29, 1, 9) & (80, 6, 41, 10, 9) \\
(0, 53, 49, 68, 84) & (6, 36, 62, 49, 0) & (53, 0, \infty, 15, 14) &
\end{array}$$

This 2-(95, 5, 1)DD contains 235 disjoint directed trades of volume 2 and 141 disjoint directed trades of volume 3.

$$\begin{array}{l}
T'_{1i} : (4 + 2i, 79 + 2i, 61 + 2i, 1 + 2i, 74 + 2i)(85 + 2i, 35 + 2i, 74 + 2i, 1 + 2i, 58 + 2i) \\
T_{1i} \\
T''_{1i} : (4 + 2i, 79 + 2i, 61 + 2i, 74 + 2i, 1 + 2i)(85 + 2i, 35 + 2i, 1 + 2i, 74 + 2i, 58 + 2i) \\
1 \leq i \leq 47; \\
T'_{2i} : (42 + 2i, 47 + 2i, 44 + 2i, 52 + 2i, 37 + 2i)(66 + 2i, 1 + 2i, 52 + 2i, 44 + 2i, 88 + 2i) \\
T_{2i} \\
T''_{2i} : (42 + 2i, 47 + 2i, 52 + 2i, 44 + 2i, 37 + 2i)(66 + 2i, 1 + 2i, 44 + 2i, 52 + 2i, 88 + 2i) \\
1 \leq i \leq 47; \\
T'_{3i} : (41 + 2i, 66 + 2i, 45 + 2i, 32 + 2i, 91 + 2i)(2 + 2i, 2i, 66 + 2i, 41 + 2i, 48 + 2i) \\
T_{3i} \\
T''_{3i} : (66 + 2i, 41 + 2i, 45 + 2i, 32 + 2i, 91 + 2i)(2 + 2i, 2i, 41 + 2i, 66 + 2i, 48 + 2i) \\
1 \leq i \leq 47; \\
T'_{4i} : (3 + 2i, 72 + 2i, 40 + 2i, 52 + 2i, 9 + 2i)(19 + 2i, 52 + 2i, 40 + 2i, 2i, 90 + 2i) \\
T_{4i} \\
T''_{4i} : (3 + 2i, 72 + 2i, 52 + 2i, 40 + 2i, 9 + 2i)(19 + 2i, 40 + 2i, 52 + 2i, 2i, 90 + 2i) \\
1 \leq i \leq 47; \\
T'_{5i} : (20 + 2i, 37 + 2i, 48 + 2i, 54 + 2i, 21 + 2i)(1 + 2i, 83 + 2i, 48 + 2i, 37 + 2i, 66 + 2i) \\
T_{5i} \\
T''_{5i} : (20 + 2i, 48 + 2i, 37 + 2i, 54 + 2i, 21 + 2i)(1 + 2i, 83 + 2i, 37 + 2i, 48 + 2i, 66 + 2i) \\
1 \leq i \leq 47; \\
R'_{1i} : (61 + 2i, 81 + 2i, 55 + 2i, 19 + 2i, 83 + 2i)(48 + 2i, 57 + 2i, 55 + 2i, 81 + 2i, 1 + 2i) \\
(81 + 2i, 57 + 2i, 5 + 2i, 84 + 2i, 17 + 2i) \\
R_{1i} \\
R''_{1i} : (61 + 2i, 55 + 2i, 81 + 2i, 19 + 2i, 83 + 2i)(48 + 2i, 81 + 2i, 57 + 2i, 55 + 2i, 1 + 2i) \\
(57 + 2i, 81 + 2i, 5 + 2i, 84 + 2i, 17 + 2i) \\
1 \leq i \leq 47; \\
R'_{2i} : (9 + 2i, 1 + 2i, 10 + 2i, 62 + 2i, 81 + 2i)(18 + 2i, 58 + 2i, 29 + 2i, 1 + 2i, 9 + 2i) \\
(80 + 2i, 6 + 2i, 41 + 2i, 10 + 2i, 9 + 2i) \\
R_{2i} \\
R''_{2i} : (1 + 2i, 10 + 2i, 9 + 2i, 62 + 2i, 81 + 2i)(18 + 2i, 58 + 2i, 29 + 2i, 9 + 2i, 1 + 2i) \\
(80 + 2i, 6 + 2i, 41 + 2i, 9 + 2i, 10 + 2i) \\
1 \leq i \leq 47;
\end{array}$$

$$\begin{aligned}
R'_{3i} : & (i, 53 + 2i, 49 + 2i, 68 + 2i, 84 + 2i)(6 + 2i, 36 + 2i, 62 + 2i, 49 + 2i, 2i) \\
& (53 + 2i, 2i, \infty, 15 + 2i, 14 + 2i) \\
R''_{3i} : & (53 + 2i, 49 + 2i, 2i, 68 + 2i, 84 + 2i)(6 + 2i, 36 + 2i, 62 + 2i, i, 49 + 2i) \\
& (2i, 53 + 2i, \infty, 15 + 2i, 14 + 2i) \\
& 1 \leq i \leq 47.
\end{aligned}$$

Therefore according to the above trades that there exist in design D , we can obtain the set of intersections of $(95, 5, 1)DD$ and can deduce $I_D(95) = J_D(95)$.

- $I_D(115) = J_D(115)$.

Let D be a $2-(115, 5, 1)DD$ on the set $V = Z_{104} \cup \{\infty_0, \infty_1, \dots, \infty_{10}\}$, that is obtained by developing the blocks below. The design D contain two groups block, the first group blocks is obtained by adding the value $+2 \pmod{104}$ to the below blocks and in this blocks ∞_0 is changed ∞_1 when adding $+2 \pmod{104}$.

$$\begin{aligned}
& (9, 61, 56, 4, \infty_0) & (61, 9, 36, 88, \infty_1) \\
& (13, 65, 60, 8, \infty_0) & (65, 13, 40, 92, \infty_1) \\
& (17, 69, 64, 12, \infty_0) & (69, 17, 44, 96, \infty_1) \\
& (21, 73, 68, 16, \infty_0) & (73, 21, 48, 100, \infty_1) \\
& (25, 77, 72, 20, \infty_0) & (77, 25, 52, 0, \infty_1) \\
& (29, 81, 76, 24, \infty_0) & (81, 29, 4, 56, \infty_1) \\
& (33, 85, 80, 28, \infty_0) & (85, 33, 8, 60, \infty_1) \\
& (37, 89, 84, 32, \infty_0) & (89, 37, 12, 64, \infty_1) \\
& (41, 93, 88, 36, \infty_0) & (93, 41, 16, 68, \infty_1) \\
& (45, 97, 92, 40, \infty_0) & (97, 45, 20, 72, \infty_1) \\
& (49, 101, 96, 44, \infty_0) & (101, 49, 24, 76, \infty_1) \\
& (53, 1, 100, 48, \infty_0) & (1, 53, 28, 80, \infty_1) \\
& (57, 5, 0, 52, \infty_0) & (5, 57, 32, 84, \infty_1)
\end{aligned}$$

The second group blocks is obtained by developing the below base blocks $+2 \pmod{104}$, and in the first block ∞_0 is replaced by ∞_1 when adding any value $\equiv 2 \pmod{4}$. Finally form a $2-(11, 5, 1)DD$, on the set $\{\infty_0, \infty_1, \dots, \infty_{10}\}$.

$$\begin{aligned}
& (0, 3, 67, 25, 17) & (6, 103, 59, 0, 33) & (0, 59, 49, 96, 28) & (0, 58, 88, 65, 84) & (0, 21, \infty_3, 93, 92) & (75, 0, \infty_5, 103, 62) \\
& (16, 67, 3, 84, 93) & (24, 87, \infty_2, 103, 6) & (83, 29, 49, 59, 12) & (53, 44, 88, 58, 78) & (92, 93, 1, 78, 89) & (0, 75, \infty_4, 64, 83) \\
& (\infty_0, 47, 2, 4, 41) & (4, 2, 26, 49, 44) & (23, 32, \infty_6, 4, 47) & (5, 3, 47, 51, 44) & (3, 5, 87, 58, 41) & (55, 2, \infty_7, 58, 87) \\
& (21, 39, 52, 3, 9) & (76, 41, \infty_9, 3, 52) & (42, 37, \infty_8, 9, 52) & (0, 4, 50, 12, 82) & (100, 44, 12, 85, 50) & (29, 94, \infty_{10}, 50, 85)
\end{aligned}$$

This $2-(115, 5, 1)DD$ contains 338 disjoint directed trades of volume 2, 208 disjoint directed trades of volume 3.

$$\begin{aligned}
S'_{1i} : & (2i, 3 + 2i, 67 + 2i, 25 + 2i, 17 + 2i)(16 + 2i, 67 + 2i, 3 + 2i, 84 + 2i, 93 + 2i) \\
S_{1i} & \\
S''_{1i} : & (2i, 67 + 2i, 3 + 2i, 25 + 2i, 17 + 2i)(16 + 2i, 3 + 2i, 67 + 2i, 84 + 2i, 93 + 2i) \\
& 0 \leq i \leq 51; \\
S'_{2i} : & (6 + 2i, 103 + 2i, 59 + 2i, 2i, 33 + 2i)(24 + 2i, 87 + 2i, \infty_2, 103 + 2i, 6 + 2i) \\
S_{2i} & \\
S''_{2i} : & (103 + 2i, 6 + 2i, 59 + 2i, 2i, 33 + 2i)(24 + 2i, 87 + 2i, \infty_2, 6 + 2i, 103 + 2i) \\
& 0 \leq i \leq 51; \\
S'_{3i} : & (2i, 59 + 2i, 49 + 2i, 96 + 2i, 28 + 2i)(83 + 2i, 29 + 2i, 49 + 2i, 59 + 2i, 12 + 2i) \\
S_{3i} & \\
S''_{3i} : & (2i, 49 + 2i, 59 + 2i, 96 + 2i, 28 + 2i)(83 + 2i, 29 + 2i, 59 + 2i, 49 + 2i, 12 + 2i) \\
& 0 \leq i \leq 51; \\
S'_{4i} : & (2i, 58 + 2i, 88 + 2i, 65 + 2i, 84 + 2i)(53 + 2i, 44 + 2i, 88 + 2i, 58 + 2i, 78 + 2i) \\
S_{4i} & \\
S''_{4i} : & (2i, 88 + 2i, 58 + 2i, 65 + 2i, 84 + 2i)(53 + 2i, 44 + 2i, 58 + 2i, 88 + 2i, 78 + 2i) \\
& 0 \leq i \leq 51;
\end{aligned}$$

$$\begin{aligned}
S'_{5i} &: (2i, 21 + 2i, \infty_3, 93 + 2i, 92 + 2i)(92 + 2i, 93 + 2i, 1 + 2i, 78 + 2i, 89 + 2i) \\
S_{5i} &: (2i, 21 + 2i, \infty_3, 92 + 2i, 93 + 2i)(93 + 2i, 92 + 2i, 1 + 2i, 78 + 2i, 89 + 2i) \\
&\quad 0 \leq i \leq 51; \\
S'_{6i} &: (75 + 2i, 2i, \infty_5, 103 + 2i, 62 + 2i)(2i, 75 + 2i, \infty_4, 64 + 2i, 83 + 2i) \\
S_{6i} &: (2i, 75 + 2i, \infty_5, 103 + 2i, 62 + 2i)(75 + 2i, 2i, \infty_4, 64 + 2i, 83 + 2i) \\
&\quad 0 \leq i \leq 51; \\
R'_{1i} &: (\infty_0, 47 + 2i, 2 + 2i, 4 + 2i, 41 + 2i)(4 + 2i, 2 + 2i, 26 + 2i, 49 + 2i, 44 + 2i) \\
&\quad (23 + 2i, 32 + 2i, \infty_6, 4 + 2i, 47 + 2i) \\
R_{1i} &: (\infty_0, 4 + 2i, 47 + 2i, 2 + 2i, 41 + 2i)(2 + 2i, 4 + 2i, 26 + 2i, 49 + 2i, 44 + 2i) \\
&\quad (23 + 2i, 32 + 2i, \infty_6, 47 + 2i, 4 + 2i) \\
&\quad 0 \leq i \leq 51; \\
R'_{2i} &: (5 + 2i, 3 + 2i, 47 + 2i, 51 + 2i, 44 + 2i)(3 + 2i, 5 + 2i, 87 + 2i, 58 + 2i, 41 + 2i) \\
&\quad (55 + 2i, 2 + 2i, \infty_7, 58 + 2i, 87 + 2i) \\
R_{2i} &: (3 + 2i, 5 + 2i, 47 + 2i, 51 + 2i, 44 + 2i)(5 + 2i, 3 + 2i, 58 + 2i, 87 + 2i, 41 + 2i) \\
&\quad (55 + 2i, 2 + 2i, \infty_7, 87 + 2i, 58 + 2i) \\
&\quad 0 \leq i \leq 51; \\
R'_{3i} &: (21 + 2i, 39 + 2i, 52 + 2i, 3 + 2i, 9 + 2i)(76 + 2i, 41 + 2i, \infty_9, 3 + 2i, 52 + 2i) \\
&\quad (42 + 2i, 37 + 2i, \infty_8, 9 + 2i, 52 + 2i) \\
R_{3i} &: (21 + 2i, 39 + 2i, 3 + 2i, 9 + 2i, 52 + 2i)(76 + 2i, 41 + 2i, \infty_9, 52 + 2i, 3 + 2i) \\
&\quad (42 + 2i, 37 + 2i, \infty_8, 52 + 2i, 9 + 2i) \\
&\quad 0 \leq i \leq 51; \\
R'_{4i} &: (2i, 4 + 2i, 50 + 2i, 12 + 2i, 82 + 2i)(100 + 2i, 44 + 2i, 12 + 2i, 85 + 2i, 50 + 2i) \\
&\quad (29 + 2i, 94 + 2i, \infty_{10}, 50 + 2i, 85 + 2i) \\
R_{4i} &: (2i, 4 + 2i, 12 + 2i, 50 + 2i, 82 + 2i)(100 + 2i, 44 + 2i, 50 + 2i, 12 + 2i, 85 + 2i) \\
&\quad (29 + 2i, 94 + 2i, \infty_{10}, 85 + 2i, 50 + 2i) \\
&\quad 0 \leq i \leq 51.
\end{aligned}$$

Therefore by help of the above directed trades that there exist in design D we can obtain the set of intersections $(115, 5, 1)DD$ and can deduce $I_D(115) = J_D(115)$.

- $I_D(v) = J_D(v)$, for $v = 135, 215, 335$.

According to Proposition 3.5 for each of values $v = 135, 215, 335$ there exists a $\{5, 6\} - GDD$ of type $5^{\frac{v-5}{10}}2$. Now by Lemma 3.3 and the fact that $I_G(10) = \{0, 1, \dots, 6, 8\}$, $I_G(12) = \{0, 1, \dots, 10, 12\}$, $I_D(5) = J_D(5)$ and $\{0, 1, 2, 3, 11\} \subseteq I_D(11)$, we can deduce $I_D(v) = J_D(v)$.

- $I_D(v) = J_D(v)$, for $v = 155, 235$.

According to Proposition 3.5 for each of values $v = 155, 235$ there exists a $\{5, 6\} - GDD$ of type $5^{\frac{v-25}{10}}12$. Now by Lemma 3.3 and the fact that $I_G(10) = \{0, 1, \dots, 6, 8\}$, $I_G(12) = \{0, 1, \dots, 10, 12\}$, $I_D(25) = J_D(25)$ and $\{0, 1, 2, 3, 11\} \subseteq I_D(11)$, we can deduce $I_D(v) = J_D(v)$.